INDIAN STATISTICAL INSTITUTE

Probability Theory I: B. Math (Hons.) I

Semester I, Academic Year 2017-18

Backpaper Exam

Date: 02/01/2018 **Total Marks: 100** **Duration: 10 am - 1 pm**

- **Show all your works and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**
- **You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. (10 marks) Suppose r balls of different colours are placed at random into n boxes arranged in a row (here $r, n \in \mathbb{N}$). Assuming that there is no limitation on the capacity of each box, compute the probability that no box in empty.

2. Consider Polya's urn scheme as described in the class. Fix $n \in \mathbb{N}$. Compute the probabilities of the following events with full justification:-
   (a) (5 marks) The n^{th} drawn ball is red.
   (b) (5 marks) The n^{th} drawn ball is black but the $(n+1)^{th}$ drawn ball is red.

3. (10 marks) Roads A and B are the only escape routes from a state prison. Prison records show that, of the prisoners who tried to escape, 40% used road A, and 60% used road B. These records also show that 80% of those who tried to escape via A, and 70% of those who tried to escape via B were captured. Suppose that two prisoners have independently and successfully escaped from the prison. What is the probability that they have used the same road to escape?

4. (10 marks) Suppose r_1 many α's and r_2 many β's are arranged at random. Let X and Y denote the numbers of α-runs and β-runs, respectively. Compute the joint probability mass function of X and Y.

5. (3+7 = 10 marks) Suppose you have n letters to be sent to n different addresses and n envelopes with the addresses written on them. Suppose you put the letters in the envelopes at random so that each envelope contains exactly one letter. Let X denote the number of letters that are put in the correct envelope. Calculate $E(X)$ and $Var(X)$.

6. (6+2+2 = 10 marks) If $X \sim Bin(m, p)$ and $Y \sim Bin(n, p)$ are independent, then find the conditional distribution of X given $X + Y$. In this case, compute $E(X|X + Y)$ and $Var(X|X + Y)$.

7. (10 marks) Suppose that the initial number N of bacteria in a bacteria colony follows Poisson distribution with parameter $\lambda > 0$. Assume that the bacteria behave independently of each other, they do not replicate and each of them die within an hour with probability $p \in (0, 1)$ independently of N. Let X denote the number of surviving bacteria in the colony after one hour. Find the probability mass function of X.

8. (10 marks) Suppose F is the cumulative distribution function of a random variable X (not necessarily discrete or absolutely continuous). Show, with full justification, that for all $u \in \mathbb{R}$,
$$\lim_{x \to u-} F(x) = P(X < u).$$

9. (2+8 = 10 marks) Assume that m and n are positive integers with $m \leq n$, and $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables following geometric distribution with parameter $p \in (0, 1)$. For all $j \in \{1, 2, \ldots, n\}$, define
$$S_j = \sum_{i=1}^{j} \ln\left(4 + X_i^3\right).$$
Show that S_m/S_n has finite mean and compute its mean.

10. (10 marks) Suppose $U \sim Unif(0, 1)$ and $Y := -\log(1 - U)$.
    (a) (5 marks) Find the probability density function of Y.
    (b) (1+4 = 5 marks) Show that Y has finite fifth moment and compute its value.